

Kant's Theory of Geometry in Light of the Development of Non-Euclidean Geometries

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With the development of non-Euclidean geometries in the nineteenth century, the concern arose as to whether these alternatives constituted a refutation of Kant's theory of geometry. Partly the concerns were related to Kant's argument that geometric judgments were *a priori* synthetic judgments, meaning that the conclusions of geometry could not be derived empirically but were yet universal principles. This aspect of universality led some to believe that the development and subsequent proof of non-Euclidean geometries implied a contradiction of Kant, whose conception of geometry was based in Euclid. In this article I will address whether or not Kant's conception of geometry can be reconciled with the conclusions of non-Euclidean geometry, and in what way Euclidean and non-Euclidean geometries can be reconciled with respect to the sensible world.

For Kant, geometric propositions can only be justified through the construction of *a priori* intuitions in the imagination, which intuitions must of necessity correspond with the sensible world: "[I]t follows that the propositions of geometry are not determinations of a mere creation of our poetic imagination, which could therefore not be referred with assurance to actual objects; but rather that they are necessarily valid of space, and consequently of all that may be found in space. . . ." (Prol. 287: 31).

There are several ways in which it is thus assumed that there is no room for non-Euclidean geometry in Kant's theory. One entails the idea that the postulates of non-Euclidean geometry cannot be conceptualized *a priori*. In addressing this concern, it is important to note the fact that non-Euclidean geometries have been proven to apply to space and the sensible world.¹ In this sense, if the intuitions of a non-Euclidean geometry are determined to be *a*

¹ To name one example, "[a]ccording to Einstein's theory [of relativity] we must expect that every two rays of light within the same plane will meet sooner or later, if their paths extend far enough" (Barker 1964, p. 50).

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priori, then there is plenty of room in Kant for the validity of such intuitions, provided that they in some way correspond to physical space.

There seems to be a natural inclination to want to jump to the conclusion that the propositions of non-Euclidean and Euclidean geometries contradict one another to the extent that if they cannot be allowed to co-exist then one or the other must be determined to be the 'true' geometry of our world. The problem is not just that Euclidean geometry can be derived soundly from its postulates, but that so too can multiple non-Euclidean geometries. Since Kant was relying on Euclidean geometry, it is assumed that there was no room in his epistemology for any non-Euclidean geometry. Yet Kant's theory on second glance actually seems to fit quite well with the possibility of alternate geometries. Take, for example, the construction of two parallel lines in my intuition. Whether or not my formal intuition of these two lines allows them to potentially intersect depends entirely on the shape which my *form* of intuition takes—that is, whether in this given instance I take space to be elliptical, spherical, or Euclidean. In each instance I will still derive what will happen with the parallel lines by *a priori* intuition alone. In point of fact, Gauss, Bolyai, and Lobachevski "all carried out their work without recourse to experiment, and thus *a priori*" (Jones 1946, p. 143). Despite this, however, we are drawn back to Kant's remark in the *Prolegomena* that "the space of the geometer is exactly the form of sensuous intuition which we find *a priori* in us, and contains the ground of the possibility of all external appearances" (288: 32). If we are to accept the possibility that both Euclidean and non-Euclidean geometries can be derived *a priori*, are we 'stuck' then when it comes to determining which one applies to external experience?

Here we come to several possibilities. One possibility, put forth by Paul Henle, is to take the division of phenomenal and physical space and argue that these are the "same space considered in different contexts, not of two separate spaces" (1962, p. 234). Another possibility, which Lucas references, is that of Ewing and Strawson who "have attempted to save Kant's account of

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geometry by maintaining that it is *a priori* true at least of phenomenal geometry—the geometry of our visual experience—that is, Euclidean” (Lucas 1969, p. 6). This desire to remedy Kant’s theory of geometry with respect to the existence of non-Euclidean geometries by the separation of space/geometry into two separate realms seems entirely unnecessary. First, Kant’s theory provides without any difficulty the sheer logical *possibility* of alternate geometries so long as the concepts of such geometries are not contradictory. “A Kantian may admit these without difficulty as being mere exercises in deduction having nothing to do with actual space. The physical use of non-Euclidean geometry is, however, another matter” (Henle 1962, p. 232). Yet why can’t both kinds of geometry be made manifest in the physical world? Taking light rays to be the physical manifestation of ‘straight lines’ can lead us, on Earth, toward accepting Euclidean geometry under certain circumstances, but in outer space the investigation of black holes would lend toward the acceptance of Riemannian geometry. Both of these possibilities can exist in the physical universe, and thus it is the circumstances of the investigation at hand that call for the application of one or another geometry, as opposed to a strict reliance on only one geometry as applicable to the physical world. The obvious concern, however, is that the only way to determine which geometry applies in what cases is empirically. Nevertheless, if we are secure that the constructions of any geometry have been derived from *a priori* intuitions, and that these constructions of geometry apply or have the potential to apply to the physical world (including the vast reaches of outer space), then there is no ground on which to say that Kant’s theory excluded the possibility of valid non-Euclidean geometries. Nor can we say that the geometry of the universe must be exclusively Bolyaian, Riemannian, *or* Euclidean.

Another reason some have mistakenly assumed that non-Euclidean geometry cannot fit within Kant’s theory is the contention that we are incapable

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of imagining (and thus intuiting *a priori*) any space other than Euclidean space.² This simply does not seem to be the case. I can imagine a globe in which the longitudinal lines intersect at the poles and yet imagine that I, standing on the face of the Earth looking upwards, imagine these longitudinal lines to run parallel over my head. In fact, Hermann von Helmholtz suggested that we could even “imagine ordering our perceptions in a non-Euclidean space” by imagining the world as reflected through a convex mirror (Grabiner 1988, p. 226). Even Escher’s drawings seem to suggest that it is quite possible to imagine a world other than the strictly Euclidean. If the figures of non-Euclidean geometry cannot be ‘drawn’ in the intuition, it further begs the question as to how Lobachevski and others were able to come up with their concepts in the first place. Following what Lobachevski perhaps assessed in his mind, I can—in an even starker example—imagine my form of intuition to be spherical and thus construct various geometrical possibilities within my intuition.

Thus I contend, following Hopkins, that we are able to both “see and picture consistently with Euclidean and non-Euclidean theories” (Hopkins 1973, p. 34). It is important here to say that while Kant, having pre-dated the development of non-Euclidean geometry, would have been functioning under the assumption of the possibility of intuiting only Euclidean concepts, I nonetheless contend that there is plenty of room to include the concepts of other geometries within Kant’s distinction between formal intuitions and the form of intuitions.

While he did not argue specifically for the possibility of more than one form of intuition, I would argue that allowing for this possibility is essential in reconciling Kant with the development of multiple non-Euclidean geometries and also with his own contention that the construction of a concept “must in its representation express universal validity for all possible intuitions which fall

² See, for example, Frege (1950), *The Foundations of Arithmetic*, Oxford: Oxford UP; and Strawson, P. F. (1966), *The Bounds of Sense: An Essay on Kant's Critique of Pure Reason*, London: Methuen.

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under the same concept” (A713/B741: 577). Since both Euclidean and non-Euclidean geometries have applicability in both intuition and the physical world, this possibility of multiple forms of intuition allows Kant’s assessment of “geometry’s unquestionable validity with regard to all objects of the sensible world” (Prol. 292: 36) to stretch into the realm of non-Euclidean geometries. This indeed meets Kant’s requirement that the form of appearance “must allow of being considered apart from all sensation” (A20/B34: 66). Here I think it deserves mention that for Kant geometric concepts aren’t given validity superficially: “It is, indeed, a necessary logical condition that a concept of the possible must not contain any contradiction; but this is not by any means sufficient to determine the *objective reality* of the concept, that is, the possibility of such an object as is thought through the concept” (emphasis added, A220/B268: 240). Where an impossibility crops up, however, is at the level of intuition (A221/B269). And thus by shifting the concept of space in the mind at the level of intuition the possibility of non-Euclidean geometries can be shown to be valid. Our form of intuition is perhaps most naturally Euclidean, as that is the geometry we are taught and with which we are most familiar. However, it is clearly possible to shift our form of intuition into a spherical or elliptical space and work with constructions within these different realms. Certainly this is what Lobachevski and Bolyai must have done, for their geometries were not entirely derived at the level of empirical observation. With this possibility of different forms of intuition in mind, the axioms of multiple geometries can thus peacefully coexist as *a priori* intuitions, but only when the form of intuition assumes the shape in the mind which best corresponds to these seemingly contradictory axioms of multiple geometries.

Jones suggested that the reason why so many people have taken the arrival of non-Euclidean geometries to imply a refutation of Kant is that such people incorrectly reason that:

- [1] Only one geometry can correctly apply to actual space.
- [2] Experience alone, therefore, can determine which geometry is true.

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[3] Kant's position that geometry is *a priori*, and independent of experience, is thus untenable. (Jones 1946, p. 139)

The problem, Jones argues, is in [1]. It seems, though, that from experience and empirical evidence we know that more than one geometry can apply to physical space. The question then becomes not which geometry applies in all cases to actual space, but rather which geometry is *necessitated by* a given instance of actual space. Thus: "Which type of geometry proves most suitable . . . depends on the type of lines we use, and the choice of the type of line for actual measurements, in turn, is affected by empirical factors. Regardless of the type of line, and thus of the type of geometry used, however, the other geometries remain sound unless it can be shown that only one type of line can be constructed in space" (Jones 1946, p. 143). We see then that those who view the development and seemingly contradictory nature of multiple geometries as a reason to insist that only one geometry can be the 'correct' one "do not fully realize that the postulates of geometry are capable of truth or falsity only when they are interpreted in some specific way. . . . What we should say is that Riemannian geometry is true when, for instance, the term 'straight line' is interpreted as meaning the path of a ray of light through a medium of uniform refractive index. . . . It is equally misleading and false to say simply that the postulates of Euclidean geometry are false—for the postulates of Euclidean geometry are true under some interpretations and false under others" (Barker 1964, p. 52).

For now, we simply can't determine *a priori* which geometry *best* describes the sensible world. In some cases appearances may deceive us as to which geometry to follow, and in such cases we must rely on empirical data to sort out our methods. However, given that there is at least one shared principle among all the geometries, i.e., that they all define a 'line' as the "shortest path between two points" [for example, in Euclidean geometry this 'line' being 'straight' and in Riemannian geometry this 'line' being an arc] (Hopkins 1973,

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p. 8), there exists the potential for a unifying theory of geometry which would someday incorporate all possibilities in a non-contradictory way.

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